Energy and Power Orthogonality in Isotropic, Discretely Inhomogeneous Waveguides

E. B. Manring and J. Asmussen, Jr., *Fellow, IEEE*

Abstract—**By analysis of the scalar potential forms of the fields, it is shown that energy orthogonality conditions for a discretely inhomogeneously-filled waveguide are actually a special case of the more general power orthogonality conditions when the fields are purely TE or TM. Power orthogonality expressions for hybrid modes may be expressed in a new form in terms of the TE and TM contributions of the *H*-field alone or the *E*-field alone. This form involves only a dot product, simplifying practical analysis when the fields are expressed in terms of TE and TM components, and clarifies the relationship between energy orthogonality and power orthogonality.**

I. Introduction

MODE ORTHOGONALITY in cylindrical waveguides containing isotropic media has been conventionally expressed in two forms, one called *energy orthogonality*, another called *power orthogonality* [1]. Derivations of each may be found in [2, pp. 389–390] and [3], respectively. Energy orthogonality does not apply when the waveguide is inhomogeneous except in the special cases, as will be shown, when the modes are TE or TM. For homogeneously-filled waveguides of arbitrary cross section, energy orthogonality is expressed as

$$\int\int \boldsymbol{E}_{t_i}^{\mathrm{TM}} \cdot \boldsymbol{E}_{t_j}^{\mathrm{TM}}\, ds = 0, \qquad i \neq j, \tag{1}$$

$$\int\int \boldsymbol{E}_{t_i}^{\mathrm{TE}} \cdot \boldsymbol{E}_{t_j}^{\mathrm{TE}}\, ds = 0, \qquad i \neq j, \tag{2}$$

where the integration is over the cross-section of the waveguide, the TE and TM superscripts refer to the scalar potential from which the fields are derived, and the t subscript indicates the transverse portion of the field. The more general power orthogonality expression for a cylindrical waveguide loaded with an inhomogeneous, isotropic material is

$$\int\int \boldsymbol{E}_{t_i} \times \boldsymbol{H}_{t_j} \cdot \boldsymbol{a}_z\, ds = 0, \qquad i \neq j, \tag{3}$$

where $\boldsymbol{a}_z$ is the axial unit vector. It is possible to show that energy orthogonality is a special case of power orthogonality when the modes are purely TE or TM. Under these circumstances, (3) may assume a form similar to (1) and (2). Furthermore, power orthogonality may be expressed in terms of dot-products of components of the E-field alone or the H-field alone.

Manuscript received November 6, 1992.
The authors are with the Department of Electrical Engineering, Michigan State University, 260 Engineering Building, East Lansing, MI 48824–1226.
IEEE Log Number 9207599.

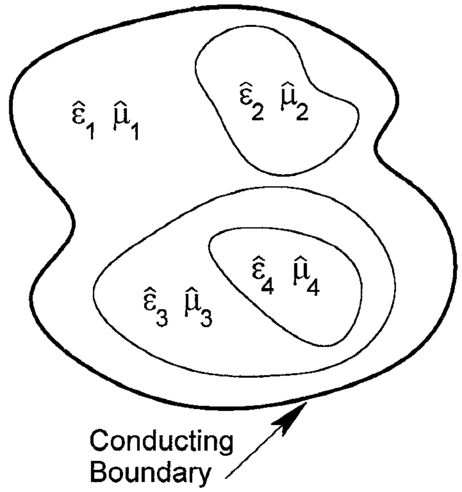

Fig. 1. Cross-section of a discretely inhomogeneously-filled cylindrical waveguide.

II. Analysis

Fig. 1 shows the cross-section of a discretely inhomogeneously-filled waveguide, uniform in the z-direction, consisting of several adjacent subregions where each subregion consists of an isotropic, homogeneous dielectric. For such a waveguide it is in general not possible to find solutions that are purely TM or purely TE to the waveguide axis. However, the fields may be constructed as a superposition of TM and TE modes.

In each homogeneous region, TM and TE solutions may be written in terms of a scalar potential, $\varphi(u,v,z) = \psi(u,v)Z(z)$, where u and v are the cross-sectional coordinates and z is the axial coordinate [2, p. 381]. Using operator notation such that

$$\nabla_t = \nabla - \frac{\partial}{\partial z}, \tag{4}$$

the constituents of the potential satisfy the following equations,

$$\nabla_t^2 \psi + k_c^2 \psi = 0, \tag{5}$$

$$\frac{\partial^2 Z}{\partial z^2} + k_z^2 Z = 0, \tag{6}$$

with

$$k_c^2 + k_z^2 = k^2 = \omega^2 \hat{\mu}\hat{\epsilon}, \tag{7}$$

where k_c is the transverse wave number, k_z is the axial wave number, and the *hat* over μ and ϵ indicates that they may be complex. The axial fields are given by

$$\boldsymbol{E}_z = \frac{k_c^2}{j\omega\hat{\epsilon}}\psi^m Z^m$$
$$\boldsymbol{H}_z = \frac{k_c^2}{j\omega\hat{\mu}}\psi^e Z^e, \quad (8)$$

where the superscripts m and e indicate a TM or a TE component, respectively. Applying boundary conditions of the continuity of $\boldsymbol{E}_z$ and $\boldsymbol{H}_z$ at the interface between homogeneous regions it is evident that Z_m, Z_e, and k_z must be the same in all regions of the waveguide. Otherwise, the boundary condition equations are z-dependent. Choosing traveling wave solutions, we write

$$Z^m = Z^e = e^{jk_z z}. \quad (9)$$

The transverse fields are then given in terms of the potentials as

$$\boldsymbol{H}_t = \left[(\nabla_t\psi^m \times \boldsymbol{a}_z) + \frac{k_z}{\omega\hat{\mu}}(\nabla_t\psi^e)\right]e^{jk_z z}, \quad (10)$$

and

$$\boldsymbol{E}_t = \left[\frac{k_z}{\omega\hat{\epsilon}}(\nabla_t\psi^m) - (\nabla_t\psi^e \times \boldsymbol{a}_z)\right]e^{jk_z z}. \quad (11)$$

Using (10) and (11), it may be shown that

$$\boldsymbol{H}_t \times \boldsymbol{a}_z = \frac{-1}{\omega\hat{\mu}k_z}\left[k^2\boldsymbol{E}_t^{\mathrm{TM}} + k_z^2\boldsymbol{E}_t^{\mathrm{TE}}\right]. \quad (12)$$

Since k_z and ω are not functions of the cross-sectional coordinates, by interchanging dot and cross products in (3), the orthogonality condition may be rewritten as

$$\int\int \boldsymbol{E}_{t_i} \cdot \frac{1}{\hat{\mu}}\left[k^2\boldsymbol{E}_{t_j}^{\mathrm{TM}} + k_z^2\boldsymbol{E}_{t_j}^{\mathrm{TE}}\right] ds = 0, \qquad i \neq j. \quad (13)$$

For certain inhomogeneously-filled waveguide configurations, non-hybrid modes, i.e., TM modes or TE modes, will propagate in an inhomogeneously-filled waveguide. An example is the ϕ-symmetric modes of a coaxially-loaded circular cylindrical waveguide. In such cases for TM modes (13) becomes

$$\int\int \hat{\epsilon}\boldsymbol{E}_{t_i}^{\mathrm{TM}} \cdot \boldsymbol{E}_{t_j}^{\mathrm{TM}} ds = 0, \qquad i \neq j, \quad (14)$$

and for TE modes

$$\int\int \frac{1}{\hat{\mu}}\boldsymbol{E}_{t_i}^{\mathrm{TE}} \cdot \boldsymbol{E}_{t_j}^{\mathrm{TE}} ds = 0, \qquad i \neq j. \quad (15)$$

Similar equations can be written for the magnetic field, i.e.,

$$\int\int \boldsymbol{H}_{t_i} \cdot \frac{1}{\hat{\epsilon}}\left[k^2\boldsymbol{H}_{t_j}^{\mathrm{TE}} + k_z^2\boldsymbol{H}_{t_j}^{\mathrm{TM}}\right] ds = 0, \qquad i \neq j, \quad (16)$$

$$\int\int \frac{1}{\hat{\epsilon}}\boldsymbol{H}_{t_i}^{\mathrm{TM}} \cdot \boldsymbol{H}_{t_j}^{\mathrm{TM}} = 0, \qquad i \neq j, \quad (17)$$

$$\int\int \hat{\mu}\boldsymbol{H}_{t_i}^{TE} \cdot \boldsymbol{H}_{t_j}^{TE} = 0, \qquad i \neq j. \quad (18)$$

For a homogeneously-filled waveguide, where only TM or TE modes exist, it is clear how these equations specialize to (1) and (2). On the basis of these equations, the cavity orthogonality conditions given by Harrington are not valid in general [2, p. 432]. The equation given for the electric field is valid only for TM modes while the equation for the magnetic field is valid only for TE modes.

III. Conclusion

Energy orthogonality in discretely inhomogeneously-filled cylindrical waveguides has been shown to be a special case of power orthogonality when the fields are purely TM or TE. Additionally, an alternative form of the power orthogonality expression has been derived. This alternative form requires knowledge of only one of the fields, E or H, albeit divided into TE and TM contributions, and involves no cross products.

References

[1] R. B. Adler, "Waves on inhomogeneous cylindrical structures," *Proc. IRE*, vol. 40, pp. 339–348, Mar. 1952.
[2] R. F. Harrington, *Time-Harmonic Electromagnetic Fields.* New York: McGraw-Hill, 1961.
[3] R. E. Collin, *Field Theory of Guided Waves*, 2nd ed. New York: IEEE Press, 1991, pp. 333–337.